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Egyptian Fractional Numerals

The grammar of Egyptian NPs and statements with fractional number expressions[1]

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Abstract

Egyptian fractional numerals are partitive expressions of two types: (a) A simple substantive specific for naming the natural fractions 'half' *gś*, 'quarter' *ḥśb*, 'third' *r*, 'two thirds' *r.wj*. (b) A complex partitive numeral formed with the substantive *r* 'part' and a number, *r*-NUM (meaning 'the *n*th-part'), which became generalized – also for 'quarter' and 'third' – as the standard Middle Egyptian denomination for fractional numbers. Fractional numerals documented in Middle Egyptian hieratic mathematical papyri are related to their argument by means of genitival syntax: *gś n r.wj* 'half of two parts'; *ꜥḥꜥ, r.wj=f* 'a quantity, two parts of it'. We propose a syntactic structure for Egyptian fractional numerals that specifies the lexical and grammatical categories and the linguistic operations used to build the numeral and its relation with the argument operated over.

1 Introduction

We study the grammar of Middle Egyptian nominal expressions containing fractional numerals as they appear documented in mathematical papyri of the 12th Dynasty (ca 1991–1786 BC), written in hieratic script. The Egyptian concept of fractional number and the linguistic means used to express fractions can be studied in the exant mathematical documents listed below in table 1. We focus our study on pRhind.

Egyptians expressed non-integer numbers by means of a system built on the notion of 'unit fraction' ('Stammbruch') (Hultsch 1985, Neugebauer 1962, Sethe 1916). Unit fractional numbers were encoded as nominal expressions of two types:

a) A simple substantive specific for naming the natural fractions[2] *gś* 'half', *ḥśb* 'quarter', *r* 'third', *r.wj* 'two parts'.

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2 Natural fractions name numbers which are immediately understood as a fractional part of a whole without performing any operation, either because they refer to a natural object made of two parts (*half*), or because a fractional part is directly perceived (*quarter*, *two thirds*, *three fourths*).

b) A compound nominal which combined the substantive *r* ‘part’ followed by a numeral denoting a number of fractional parts:

(1) *r-djw*
part(M)[SG]-5(UF)
‘fifth part’

	name	date	script	edition
pReisner	Reisner Papyrus I	11 Dyn (2125–1985)	hieratic	Simpson (1963)
wtAkhmin	Akhmim Wooden Tablet	1991 BC	hieratic	Daressy (1906)
pMoscow	Moscow Mathematical Papyrus	ca. 1789 BC	hieratic	Struve (1930)
pKahun	Kahun Papyri	Hyksos ca. 1700	hieratic	Griffith (1898)
pRhind	Ahmes Mathematical Papyrus	ca. 1650. Copied by Ahmes from 1850 ms (Hyksos)	hieratic	Chace (1927–1929), Eisenlohr (1877), Peet (1923)
lrM	Egyptian Mathematical Leader Roll	Hyksos	hieratic	Glanville (1927)
pDemotic	Demotic Mathematical Papyri	Greco-Roman period	demotic	Parker (1972)
pEbers	Ebers Papyrus	1550	hieratic	Ebers (1875)

Table 1: Mathematical Papyri

The construction “*r*-NUM” was the general notation used for fractional numbers. Such a uni-number partitive numeral expressed a quotient, understood as a non- integer unit which was the last part of a whole divided into a number *n* of equal parts (Gardiner 1957: §126). In mathematical papyri, fractions were written in hieratic script: *gś* ‘half’, *r:wj* ‘two parts’, *r-ẖmtw* ‘third’, and *r-fdw* ‘quarter’ were represented by specific symbols; other unit fractions were expressed by a cardinal with a dot over it.

Egyptians expressed mixed fractions, with a numerator ⩾ |2|, as the sum of two or more unit fractions (*ỉ3t* ‘group of fractions’) with different number of parts, the only exception being ⅔, which was a basic one in calculations, and ¾ used in length measurement. As an illustration, the non-integer number yield from dividing 2 by 5 was expressed as a sum of two unit fractions $\dot{3} + \dot{1}5$ (⅓ + ⅟15), and not by a single fraction ⅖. Furthermore, because the same fraction could not be repeated, ⅖ was not expressed as $\dot{5} + \dot{5}$ (⅕ + ⅕).

Fractions such as ⅔, ¾ have been named “Komplementbrüche” (‘complementary fractions’) (Sethe 1916. III.6: 91–103) because they refer to the parts which, added to a unit fraction, make a whole. Like unit fractions, complementary fractions are also formed with the substantive *r*, but with dual (⅔) or plural (¾) morphological number (Loprieno 1986: 1307):

(2a)	*r.wj*		(2b)	*ẖmtw*	*r.w*
	part.DU(CF)			3	part.PL(CF)
	'2 parts (of 3)'			'3 parts (of 4)'	

A "complementary fraction" such as *r.wj* was understood as 'the two parts', which complete a total of 3 fractional parts (Edel 1955, Gardiner 1957, Sethe 1916). Complementary fractions were, thus, bi-number expressions with the total number of parts left implicit.

It has been shown that the concept of fraction denoted by Egyptian fractional numbers and fractional words evolved from a metrological meaning of 'part of a measuring unit' onto a partitive concept of 'fractional-part' and later onto a number concept of 'fractional-part unit' (Clagett 1999, Ritter 1992, Sethe 1916). Sethe (1916) states that Egyptian metrological fractions were built on some primeval idea of 'side' of the body of an animal (*rmn ỉḥ*, 'side of a cow') or of a human being (*gś rmṯ*, 'side of a man'), which became extended in metrological systems to denote some part (*rmn*, 'upper arm' unit) of an extension of land (*mḥ*, 'cubit' unit), or a 'share' (*psš.t*)[3] in some distribution of food or goods: i.e. some kind of 'mouth- full' share (*r*). Such an extension of the notion of side-part-share manifested itself both in the words used to express the 'part' idea, and in the partitive grammatical construction encoding Egyptian fractional numbers (Sethe 1916).

- How did metrological fractions become to denote the class of fractional numbers?
- What procedure was used to generate an infinite set of fractions from natural fractions?

The expansion of the idea of 'fractional-part' onto the concept of 'fractional number' came from reckoning processes (*tp ḥśb*)[4] in which unit fractions were applied to solve many different problems. For instance, problems of division (*psš*) returning equal (pRhind problems 1-6) or proportional shares (pRhind problems 63, 64, 65; 39, 40, 11), or problems for reckoning the productivity (*bꜣkw*) of workers (pMoscow 23, 11). Egyptian scribes showed a great concern for fractions, evidenced by the large proportion of problems involving operations with fractions. Unit fractions are prevalent in extant mathematical documents. For instance, in pRhind, all but problems number 48, 62, 77–79 use fractions for their calculations. Moreover, a 2:n table in Rhind contains divisions of 2 by odd numbers from 5 to 101 expressed as a sum of unit fractions. In lrM there is a table of multiplication of fractions. In the medical papyrus Ebers, the proportion of ingredients is expressed by means of Eye of Horus fractions written in hieratic script. One reason why fractions were so important for Egyptian mathematicians could be that the fraction idea

3 The hieratic notation for 'share' is documented in pRhind 64: *psš.t mtr.t* 'average share'. Erman & Grapow (1940) transcribe the word meaning 'to divide' as *pśš*. Gardiner (1957: 566, 538) transcribes it as *psš*. and 'division, share' as *psšt*. In this paper we use Gardiner's transcription. For the hieratic notation in pRhind cf. table 7.

4 The operation of calculating or reckoning by means of reiterated breaking into halves (halving) seems to have been expressed with the word *ḥśb* (Gardiner 1957: 582). For reckoning processes based on a counting operation, it was used the word *ỉp* (Morenz 2013: §2; Gardiner 1957: 553). The idea of 'number' was encoded by means of the abstract noun *ỉp.t* (Gardiner 1957: 34; Morenz 2013: 23), and in Demotic also by the DP *pꜣ nkt* (fn 17, ex. ib), or by *pꜣ r* (cf. ex. 52).

constituted a natural operation which played a core role in mathematics. The procedure of fractioning, understood as ‘breaking’ in halves, was a fundamental operation. The term for ‘breaking’ *ḥśb* was used both to denote an action of ‘calculating’, ‘reckoning’, and also to name a unit fraction 4̇, ‘quarter’ obtained by a dimidiating operation over the natural fraction *gś* ‘half’. Halving was further applied to the other natural fraction 3̈, ‘two parts’ which returned one of such ‘parts’ 3̇, originating the ‘fractional part’ *r* idea. And the procedure of halving was applied extensively to any number.

Such a prevalent use of the fractional part idea applied to problem resolution expanded metrological fractions onto a fractional number sequence and hence, contributed to creating the class of rational numbers. However, the core idea of a fractional-part – expressed as a unit fraction numeral – was not abandoned, and non-unit fractions were expressed as a sum of unit fractions with an equivalent quantitative value; i.e. a group of fractions *ỉꜣt*. Those fractional expansions that came into being from reckoning processes (*tp ḥśb*) have been named by Neugebauer (1962) “algorithmic fractions”. The nature of those algorithms has been the focus of scientific study of all times.

The aim of our work is to study:

- The lexical and grammatical categories encoding Egyptian fractional numerals.
- The syntactic operations used for building fractional numerals and their relation with the argument over which they operate.

In this article we propose a syntactic structure for nominals with fractional numerals. Our linguistic analysis is grounded on Gardiner (1957), Kammerzell (2000), Loprieno (1986, 1995), Schenkel (1963, 1990), Sethe (1916). We draw the mathematics of Egyptian fractions from Neugebauer (1962). In section 2 we contrast the use of fractions as subdivisions of metrological units and as a number concept. In section 3 we focus on the lexical and syntactic categories that the Egyptian language used to encode the concept of fractional numbers.

2 Metrological fractions and fractional numbers

2.1 Fractions as subdivisions of metrological units

The oldest fractional expressions did not appear as referentially independent numbers, but they were associated with units in metrological systems (Ritter 1992). Metrological units and their divisions were dubbed with names of parts of the body, natural objects or containers which were taken as models for measuring units with some fixed value.[5]

The Palermo Stone is one of the sources for the early fraction symbols. There appear hieroglyph signs for fractions ⅔, ¾, ½ associated to length measurement units.[6] The

5 In the length system, ‘royal cubit’ *mḥ nsw*, ‘forearm, cubit’ *mḥ* (7 palms, 28 digits), ‘upper arm’ *rmn* (5 palms, 20 digits), ‘palm’ *šsp* (4 digits), ‘hand’ *ḏrt* (5 digits), ‘digit’ *ḏbꜥ*, ‘rod of cord’ *ḫt-n-nwḥ* (100 cubits), or ‘river’ *ỉtr* (20,000 cubits).

6 Möller (1909–1936: vol II: 667–681: Palermo Stone RS 2, 2.3: Pellegrini (1895: 583); Schäeffer (1902: 483); Sethe (1916: 81–83); Edel (1955: §410:178); Wilkinson (2000); Morenz (2013: 25–28).

Palermo Stone is an inscription recording the Royal Annals of the first five dynasties of Memphis. A biennial survey of the wealth of the Old Kingdom is specified by counting (*ṯnwt*) (Morenz 2013: 23). The inscription is divided into registers, each with a series of boxes. In each box there is a report of the major events of the reign of a Pharaoh (military campaigns, religious ceremonies, donations). The level of the maximum height reached by the Nile river for that year, measured in cubits *mḥ*, palms *šsp*, and fingers *ḏbᶜ* appears recorded framed by a rectagle below each box. In table 2 we represent hieroglyph symbols of the fractions engraved in the Palermo Stone.

glyph	cipher	location	dynasty
	⅔	PS r V box 3: 3 2/3 cubits (Wilkinson 2000: 133)	2nd
	¾	PS r VI box 4: 2 cubits, 2 palms 2 3/4 fingers (Wilkinson 2000: 144)	4th
	½	PS v II box 2: 4 cubits, 3 palms, 2 1/2 fingers (Schäefer 1902:32–22; Wilkinson 2000: 153)	5th

Table 2: Fractions in the metrological length system recorded in Palermo stone

The units of the length system were recorded in metrical rods which took the cubit *mḥ* as a basic unit (Lepsius 1865).[7] Metrological fractions of the royal cubit (Möller: 679) *mḥ nsw*, with a value equivalent to ~52.3 cm in Old Kingdom and Middle Kingdom (Pommerening 2013), are represented in table 3. The finger length was further subdivided up to 16 parts. In scaled metrical rods, each of the first 15 fingers was consecutively divided in 2, 3,...,16 parts (Lepsius 1865: 43–44; Petrie 1926).[8]

	transliteration	value
	*mḥ nsw*	'royal cubit' = ~ 52.3 cm = 1/100 *ẖt*
	*šsp*	'palm' = ⅐ of a royal cubit
	*ḏbᶜ*	'finger' = ¼ of a palm = 1⁄28 of a royal cubit

Table 3: Parts of a royal cubit *mḥ nsw*

7 For larger measures there were used multiples of the cubit: the "khet"-unit (*ẖt-n-nwḥ*; 1 *mḥ nsw* 'royal cubit' = 1/100 *ẖt*) was used for field measurements, and the "river"-unit *ỉtr*, which was the longest measure, was used for itineraries by boat on the Nile (Pommerening 2013).

8 Consecutive marking in scaled rods evidences an action of counting the parts obtained from division. The ordinal denomination in partitive numerals with the form "r-num" 'the nth part' had its roots in counting: The ordinal number component in partitive numerals meant 'sequential counting' and not 'an ordered sequence of parts'.

In addition to the fractions ⅔, ¾, ½ (Royal Annals) and the fractions ½, ⅓, ¼,..., 1⁄16 (length measurement), other sequences of fractions used in metrological systems were the ones subdividing the area *śṯꜣt*-unit[9] and the volume *ḥḳꜣt*-unit[10] (Loprieno 1986). Area and volume fractions were obtained by recursive halving, which generated a sequence of fractions in geometric progression with ratio ½: In the area system there were 3 fractions (½, ¼, ⅛), and in the volume there were 6 (½, ¼, ⅛, 1⁄16, 1⁄32, 1⁄64).

Special names were used to label fractions of a *śṯꜣt*-unit (Sethe 1916: 72–81; Gardiner 1957: §266; Loprieno 1986: 1307; Pommerening 2013). In table 4 we represent fractions of a *śṯꜣt*.

	transliteration	value
	*śṯꜣt*	1 *ḫt*^{2} = 10,000 cubits^2
	*rmn*	½ *śṯꜣt* = 5,000 cubits^2
	*ḥśb*	¼ *śṯꜣt* = 2,500 cubits^2
	*sꜣ*	⅛ *śṯꜣt* = 1,250 cubits^2

Table 4: Fractions of a *śṯꜣ*

Of those are particularly important *rmn* and *ḥśb*, which were the primitive terms used to name fractions 'half' and 'quarter':

(3a) *rmn n ỉh*
side P animal
'side of an animal'

(3b) *ḥśp n ḫt n mw*
break of rod of cord
'quarter of a rod of cord'
(Gardiner 1927: parr. 266:199)

The word *rmn* (Möller 687) denoted the natural fraction 'half' by referring to one of two sides of a body. It was also used as a metrical unit with the size of a man's upper arm. The word *ḥśb* (Möller: 688) – late variant *ḥśp* (Fecht 1985: 85–89) – described an action of 'breaking', which when applied to partitioning the natural fraction *rmn* 'half' it returned a 'quarter' fraction.

A unit for the volume system was the *ḥḳꜣt* (ca. 4.8 liters), used for barley, wheat, corn and grain. For smaller quantities a *hnw*-unit ('jar', ca. 0.48 liters, 1⁄10 of a *ḥḳꜣt*) was used (pRhind 80):[11]

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9 The Egyptian equivalent to the greek aroura, "acre".

10 The Egyptian "bushel" (Peet 1923: 5), "Scheffel" (Eisenlohr 1877; Sethe 1916), "galon" (Gillings 1982: 210).

11 For measuring the content of large grain vessels, the unit could be either double-*ḥḳꜣt* or quadruple-*ḥḳꜣt* (pRhind 68).

transliteration	value
ḥḳꜣt	1 *ḥḳꜣt* = ~4,800 cm^3
hnw	1 *hnw* = 1⁄10 of a *ḥḳꜣt*, ~480 cm^3
r	1 *r* = 1⁄320 *ḥḳꜣt*
	5 *r* = 1⁄64 *ḥḳꜣt*

Table 5: Volume units

The *ḥḳꜣt*-unit was subdivided in 6 halving fractions ½, ¼, ⅛, 1⁄16, 1⁄32, 1⁄64, which were written using special hieratic symbols from the Eye of Horus notation (pRhind 47, 68–70) (Gardiner 1957: 197–199, note 4). Fractions below 1⁄64 of a *ḥḳꜣt* were indicated in terms of a r^c-measure (r^c = 1⁄32 x 1⁄10 = 1⁄320 of a *ḥḳꜣt*),[12] up to 5 r^c units (5r^c = 1⁄64). Special hieratic symbols were used to represent 1r, 2r, 3r, 4r.

(4) Fractions of a *ḥḳꜣt* pRhind 69

(4a) 

½ ¼ ⅛ 1⁄16 1⁄32 1⁄64

(4b) 

r 2r 3r 4r

In sum,

a) Fractions in the area and volume systems were sequentially generated by means of the Egyptian technique of division by recursive dimidiation. Also, division by 10 or multiples was used (r^c = 1⁄360 *ḥḳꜣt*). Other fractions were obtained by simple partition and sequential enumeration of parts, as with the parts of a *ḏb*^{c} 'digit' (½, ⅓, ¼…1⁄16).

b) Fractions used in metrological systems were dependent on the units they divided. Such a dependency manifested itself on the following features:
   - Metrical fractions were finite in number. Moreover, each metrical system had a different number of fractions. The area system had a sequence of 4 fractions with ratio ½: ½,¼,⅛,1⁄16. The capacity a sequence of 6: ½,¼,⅛,1⁄16, 1⁄32,1⁄64, and 5 smaller r^c fractions 1r^c 1⁄320 through 5r^c 1⁄64. The length system scaled the finger in 16 subdivisions ½,⅓,¼…1⁄16.
   - The argument divided by a fraction was a constant, referring to a unit with a fixed value in a particular metrological system. Therefore, each fraction in every particular system referred to a constant value.
   - There was no general notation for fractions. The same fraction was represented by different symbols in the area (fractions of a *śṯꜣt*) and volume (fractions of a *ḥḳꜣt*, fractions of a r^c) systems.

12 The r^c was an amount between a dessertspoon and a tablespoon (Gillings 1982: 210).

How did fractions become to refer to a number concept not dependent on metrological units? We address that question in next section.

2.2 Fractions as an independent number concept

What procedure triggered the origin of fractional expressions as a class of numbers? In Neugebauer's opinion, fractions occurred when division – applied to a variable –performed by successive halving left a remainder (*ḏꜣt*). Neugebauer illustrates such a consideration by means of a division operation of 16 by 3 (Neugebauer 1962: 74). To divide 16 by 3 in the Egyptian way you are to find how many times is 3 in 16, with counting starting with 3. The number of times is computed through successive doubling of the initial number 3. The scribe would write in one column the number of doublings, and in another the numbers doubled, starting form the divisor (figure 1):

	Operation			
initialization	$\boxed{1} \times 3$	$\boxed{\mathbf{3}}$	$\leftarrow$ start with divisor	
doubling	2×3	6		
doubling	$\boxed{4} \times 3$	$\boxed{12}$	$\rightarrow \boxed{15}$	dividend
quotient	$\boxed{5}$			remainder:1

Figure 1: The Egyptian way of dividing 16 by 3

The division of 16 (dividend) by 3 (divisor): get 16 by operating on 3: The doubling of the divisor 3 is repeated twice, when it reaches 12, the nearest number to 16. Both 3 and 12 are added and subtracted from 16 (dividend) leaving a remainder of 1. The quotient is the sum of the initial (1) and the last (4) of the number of doubling operations (5) plus the remainder 1. The division of the remainder 1 by 3 (getting 1 by operating with 3) yields as a quotient the fraction ⅓:

$1 \div 3 = ⅓$

However, the scribe would not proceed directly from 1:3 to ⅓, but ⅓ would be obtained from its double ⅔: The scribe would get first ⅔ of 3 (⅔ of 3 = 2) and then half of ⅔ (½ of ⅔ = ⅓) (figure 2):

$$\begin{array}{c|c} \ddot{3} & 2 \\ \boxed{\dot{3}} & 1 \end{array}$$

Figure 2: The division by 3 of the remainder 1

The scribe would thus get the result: a quotient expressed as the sum of a whole number 5 and a fraction of a whole number ⅓:

$16 \div 3 = 5 + ⅓$

In sum, the fraction idea was triggered by the need to express a non-integer quotient. Egyptian fractions symbolised a quotient that could not be stated by an integer (Hultsch 1985). Metrological fractions started to be numbers when the argument of a fractioning operation was a variable number rather than a constant value of a unit of measurement.

What were the linguistic statements expressing the Egyptian way of performing division? We will address that question in next section.

2.3 Multiplication and division with fractions

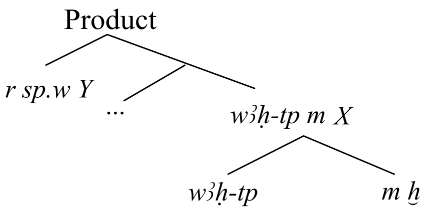
In this section we study linguistic statements of division which show the technique used by the Egyptians to perform that operation. Because division was done as the revers of multiplication, we start reviewing statements of multiplication. The examples are drawn from pRhind, and some from pDemotic. The terms are taken from Chace (1927–1929), Erman & Grapow (1940), Parker (1972), Peet (1923).

2.3.1 Multiplication

Multiplication was done by reiterated doubling.[13] The procedure is expressed by a construction "*wꜣḥ-tp...sp* NUM" 'nodding the head with a number so many times'.[14] A multiplier number phrase "*sp* NUM", '*n*-times', states the number of repetitions:

(5a)	*wꜣḥ-tp*	*m*	*X*	*sp*	*Y*
	count	with	NUM	times	NUM
	'multiply *X* *Y*-times'				

(5b)



The example below illustrates a multiplication statement:

(6)	pRhind 44								
	wꜣḥ-tp	*m*	*100*	*r*	*sp.w*	*10*	*ḫpr.ḫr=f*	*m*	*1000*
	count	with	100(CARD)	up_to	times	10(CARD)	become.RES=3MSG	P	1000
	'Multiply 100 by 10; result 1000'								

13 Multiplication is direct only with 10, 100 or with 2. Multiplication by 2 is an act of memory reproducing the results registered in tables.

14 The verb *wꜣḥ-tp* 'to nod the head' may refer to gestures done when counting either by 1, or else by every 5 or 10, counted off on the fingers. In pRhind 26, 44 and 60 it is used also *wꜣḥ* without *tp*.

Demotic uses the verb *ỉr* ‘to make’ instead of *wꜣḥ(-tp)*:

(7) pDemotic 42
ỉw=k *ỉr* *2* *r* *sp* *2* *r* *4*
PCL=2SG make 2(CARD) to times 2 result 4(CARD)
‘Multiply 2 up to 2-times; it results 4.’

Table 6 summarises multiplication terms.

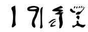
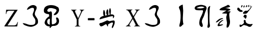
	*ỉr*	V	‘to do’
	*tp*	N	‘example’, ‘an instance of’
	*wꜣḥ*	V	‘to nod the head’
	*wꜣḥ-tp*	V	‘to count instances while nodding the head’
NUM 	*sp* NUM	N	multiplicative numeral, ‘*n* times’
	*ḫpr*	V	‘to become’
	*wꜣḥ-tp m X sp Y ḫpr m Z*	S	’count with *X* *Y*-times: becomes *Z*’

Table 6: Multiplication terms in pRhind

2.3.2 Division

Division was done as the revers of multiplication. The verbs used in division statements were: *nỉś* ‘to call’, ‘to summon’, *wꜣḥ-tp* ‘to nod the head’, ‘to count’, or the more general verb *ỉr* ‘to make’, combined with the prepositions *ḫnt* ‘out of’, ‘from among’, *r* ‘to’, *m* ‘with’; i.e. ‘call a dividend number X by operating Z-times on a divisor number Y’. The constructions were:

(8a) *nỉś* *X* *ḫnt* *Y*
call NUM out_of NUM
‘Divide *X* by *Y*.’

(8b) *wꜣḥ-tp* *m* *Y* *r* *gm.t* *X*
count with NUM to finding NUM
‘Divide *X* by *Y*.’
‘Count how many times is *Y* in *X*.’

(8c) Quotient
r gm.t X *wꜣḥ-tp m Y*
wꜣḥ-tp *m Y*

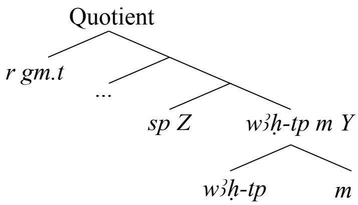
As with multiplication, the number of operations needed to be performed to reach a number was expressed by the multiplier phrase “*sp* NUM”.

(9a) pRhind 30

wȝḥ-tp	*m*	*Y*	*sp*	*Z*	*r*	*gm.t*	*X*
operate	with	NUM	times	NUM	to	finding	NUM

‘Operate with *Y* *Z*-times to find *X*.’

(9b)



The examples below illustrate division statements:

(10a) pRhind ⅔n table

nìś	*2*	*ẖnt*	*5*
call	2(CARD)	out_of	5(CARD)

‘Divide 2 by 5’

(10b) pRhind 21

wȝḥ-tp	*m*	*15*	*r*	*gm.t*	*4*
count	with	15(CARD)	to	finding	4(CARD)

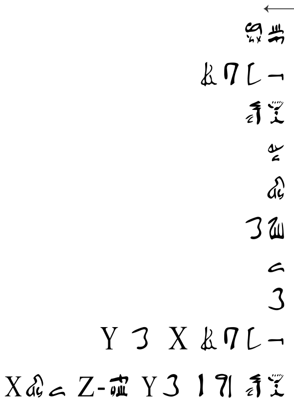
‘Divide 4 by 15’

(10c) pRhind 30

ìr.t	*3̇0*	*sp*	*2̇3*	*r*	*gm.t*	*3̈*	*1̇0*
making	30(UF)	times	23(UF)	to	finding	2(CF)	10(UF)

‘The making of 1⁄30 1⁄23-times for the finding of ⅔ + 1⁄10.’

Table 7 summarises division terms.



psš	V	‘to distribute’
nìś	V	‘to call’, ‘to summon’
wȝḥ	V	‘to nod the head’
ìr	V	‘to make’, ‘to do’
gm	V	‘to find’
ẖnt	P	‘out of’
r	P	‘to’
m	P	‘with’
nìś X ẖnt Y	S	‘Call *X* out of *Y*’
wȝḥ-tp m Y sp-Z r gm.t X	S	‘Count with *Y* *Z*-times to finding *X*’

Table 7: Division terms in pRhind

pDemotic used instead of *wꜣḥ*-tp, the verbs: *fy* ‘to carry’, *ṯꜣ* ‘to take’, *pš* ‘to apportion’:

(11a) pDemotic 61

fy	*35*	*r*	*39*	*r*	*1*	*r-10*	*r-70*
carry	35(CARD)	to	39(CARD);	result	1(CARD)	10(UF)	70(UF)

‘You shall carry 35 into 39; result 1 + 1/10+1/70.’

(11b) pDemotic 3:5

ỉw=k	*ṯꜣ*	*pꜣ*	⌜*rꜣ*⌝	*47*	*n*	*100*
DIR=2SG	take	ART.M	⌜number(M)⌝	47(CARD)	to	100(CARD)

‘Divide 100 by 47.’

The class of fractional numbers was expanded onto an infinite set through their use as arguments in arithmetic operations applied to solve many problems.

2.3.3 Applications of division with fractions

Applications of division involving the use of fractions include: (a) Problems dealing with distributional operations returning equal (pRhind 1–6) or proportional (pRhind 39, 40, 63, 64, 65) shares: *psš*-problems. (b) Problems reckoning (*ḥśb*) the productivity of workers: *bꜣkw*-problems.

We focus on the operations with fractions in problems of distribution of loaves of bread in equal shares illustrated by pRhind problems 1–6. The method used to solve such problems was worded as “*tp n psš*”, ‘example of distributing’ or ‘of dividing in shares’.

(12) pRhind 1

tp	*n*	*psš.t*	*tꜣ.w*	*n*	*s*	*10*
example	of	distributing	loaves(M).PL	among	man	10(CARD)

‘Example of the distribution of loaves of bread among 10 men.’

In each of the problems the scribe gives first his answer, which is identical to the quotients of division of 1, 2, …, 9 by 10 listed in a “*n*/10” table (table 8 below) located after the “2/*n*” table. After, the scribe proves that his answer is a correct one by performing a sum of parts, which are added using the technique of recursive duplication of the quotient until reaching the divisor number of 10 shares.

Dividend *nỉś* NUM	Divisor *ḫnt 10*	Quotient NUM	pRhind
1		/10	1
2		/5	2
3		/5 + /10	
4		/3 + /15	
5		/2	
6		/2 + /10	3
7		2/3 + /30	4
8		2/3 + /10 + /30	
9		2/3 + /5 + /30	5

nỉś-ḫr=k *7* *ḫnt 10* *m 2/3 1/30*
call-DIR=2SG 7 out_of 10 P 2/3 1/30
'Call 7 out of 10; result 2/3 + 1/30'

Table 8: *n*/10

We illustrate the operation of distribution in equal shares with pRhind 4. In this problem 7 loaves of bread are to be distributed in 10 shares each one with one portion of ⅔ of a loaf plus another of 1⁄30 of a loaf.[15]

(13a) pRhind 4:1
tꜣ.w *7* *n* *s* *10*
loaf(M).PL 7(CARD) for man 10(CARD)
'(the distribution of) 7 loaves of bread among 10 men.'

(13b) pRhind 4:1–2
ỉr.ḫ[r]=k *m* *3̈* *3̇0* *sp* *10* *ḫpr(r).ḫr* *7*
make.DIR=2SG P 2(CF) 30(UF) times 10(CARD) become.RES 7(CARD)
'Multiply 2/3 1/30 by 10; the result is 7.'

Proof by testing. To check the division operation was correct, the fractions corresponding to each share of the divided 10 loaves were added by the method of successive doubling up to 10 times (1, <u>2</u>, 4, <u>8</u>), starting with the quotient ⅔ + 1⁄30:

15 From the table 1-9 divided by 10, we get the quotient expressed as a sum: 7 divided by 10 yields 2/3 + 1/30. Gillings (1982:123) sugests 7/10 is obtained as (8-1)/10.

(14) Proof: sum of parts

ỉr.t *mỉ*
doing same
'Doing the same calculation'
('Mutiply 2/3 + 1/30 by 10')

[1]	(share is)	$\ddot{3}$ (+) [$\dot{30}$]
\ [2	(shares are)	$\boxed{1]\ \dot{3}\ (+)\ \dot{15}}$
4	(shares are)	2 $\ddot{3}$ (+) $\dot{10}$ (+) $\dot{30}$
\ 8	(shares are)	$\boxed{5\ \dot{2}\ (+)\ \dot{10}}$

dmḏ *tꜣ.w* *7* *n.t* *pw*
total loaves 7; it is this
'Total sum: 7 loaves. This is correct.'

Division did not perform equal partitions in each loaf of bread (i.e.: each loaf divided into 10 parts; each men gets ⅒ of each of the 7 breads: 7(⅒)). There were two divisions: (a) 7 loaves are divided in ⅔: there result 10 ⅔-portions of a loaf each, and ⅓ of one loaf remains. (b) The remainder ⅓ of the last loaf is divided in 10 parts (becoming 1⁄30 of the loaf), one for each man. So each men receives 2 portions of a different size: a large portion of a size of ⅔ of 1 loaf and a tiny portion of a size of 1⁄30 of a loaf (fig 3).

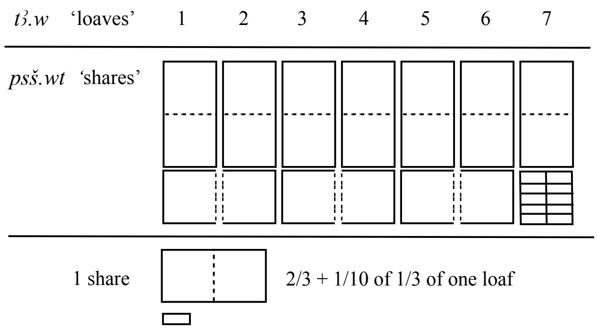


Figure 3: The fractioning of 7 loaves of bread in 10 shares

Because such divisions with fractions were complicated, Egyptian scribes made tables to ease the calculations. The use of fractions for those calculations contributed to creating the class of fractional numbers.

2.4 Properties of Egyptian fractions as a class of numbers

The use of fractional expressions as numbers is documented in Middle Egyptian mathematical texts. The number status of fractional expressions is manifested in the following properties: (a) The use of a general notation to express fractions; (b) Self-reference; (c) Fractions form a densely ordered number sequence; (d) Fractions take part as arguments in the general arithmetic operations applying to numbers.

(a) General notation. Egyptian Fractions are ciphered by means of a general notation related to the class of cardinal numbers. In mathematical texts, fractions were written in hieratic script: ⅔, ⅓, ½ and ¼ were represented by a particular sign inherited from metrological fractions. Other unit fractions were represented by the same ciphers used for cardinals with a dot above – standing for the word *r*: $\dot{5}$, $\dot{7}$.

(15) Natural fractions in hieratic notation

r.wj	*gś*	*r-ẖmtw*	*r-fdw*
$\ddot{3}$	$\dot{2}$	$\dot{3}$	$\dot{4}$
2(CF)	2(UF)	3(UF)	4(UF)
⅔	½	⅓	¼

(16) Unit fractions in the general hieratic notation

(16a)

r-djw	*r-sjsw*	*r-sfẖw*	*r-ẖmnw*	*r-psḏw*
$\dot{5}$	$\dot{6}$	$\dot{7}$	$\dot{8}$	$\dot{9}$
5(UF)	6(UF)	7(UF)	8(UF)	9(UF)
⅕	⅙	⅐	⅛	⅑

(16b)

r-mḏw	*r-mḏw.tj*	*r r-š(n)t*	*r-š(n)t.j*
$\dot{1}0$	$\dot{2}0$	$\dot{1}00$	$\dot{2}00$
10(UF)	20(UF)	100(UF)	200(UF)
1⁄10	1⁄20	1⁄100	1⁄200

(b) Self-reference. Egyptian fractions denote a world independent property, dissociated from some metrological dimension. The dividend argument of a fraction is an open position to be filled by a value in the domain of numbers and not by some metrological unit. Fractions are thought of as numbers which may be referred to as an unknown quantity *pꜣ ꜥḥꜥ* (pRhind 30). The self-referring property of fractions is fully evidenced in pDemotic, where the DP *pꜣ r* is used to denote both 'the fraction' and 'the number' (cf. footnote 17, and example 52).

(c) A dense number sequence. The elements in the class of fractional numbers are ordered in a dense linear number sequence. Between any two fractions there are infinite many fractions:

(17) Dense Ordered Set.
An ordered set $(D, <)$ is *dense* if it has at least two diverse elements x, y and if for any $x, y \in D$, $x < y$ implies that exists $z \in D$, diverse from x, y, such that $x < z$ and $z < y$.[16] More formally expressed:

$$D_< = \forall x, y \in D \, [\, x < y \rightarrow \exists z \in D \, [\, x < z \wedge z < y \,]]$$

However, Egyptian fractions with numerator > |1| were expressed as some equivalent series of sum of unit fractions, *ꜣ͗t* 'group of fractions' (Peet 1923: 77. pRhind 38). The examples below are from the $2/n$ table in pRhind:

(18) (a) $2/5 = 1/3 + 1/15$ (c) $2/25 = 1/15 + 1/75$
(b) $2/7 = 1/4 + 1/28$ (d) $2/31 = 1/20 + 1/124 + 1/155$

Only the double of $1/3$ was represented by a single number *r.wj* $2/3$, which was taken to denote a natural fraction. But the double of a fraction with an odd number of parts was not symbolised by a simple fractional number ($2/5$, $2/7$...$2/101$). Those were encoded as "algorithmic fractions" (Neugebauer 1962), expressed by some sum of unit fractions equivalent to the doubled one.

(d) Arithmetic operations with fractions. In Mathematical Papyri, fractions are used as argument of the general arithmetic operations that apply to numbers. Egyptian arithmetics used addition, subtraction, and the techniques of doubling and halving a quantity, taking $2/3$ of it, multiplying it by 10, and taking $1/10$ of it, and finding fractional multipliers by the use of reciprocals (Clagett 1999: 18). The extension of the sequence built on natural fractions onto a dense linear number sequence was a natural consequence of the widespread use of fractions in operations applied to solve a large number of problems.

3 The grammar of fractional numerals

In this section we study the syntactic structure of Middle Egyptian nominals with fractional numerals denoting a unit fraction (*gś*, *r*-NUM) and a complementary fraction *r.wj*. In section 3.1 we contrast the lexical and selectional properties of those numerals and propose a derivation in which their lexical and grammatical features are merged. In section 3.2 we focus on the syntactic structure that projects the operation of a fractional numeral over its argument.

3.1 The fractional numeral

Unit fractions expressed by *gś* 'half' and by the general fraction numeral "*r*-NUM" formed with the substantive *r* 'part', and the complementary fraction with the form *r.wj* 'two parts' are nominal categories. The simple fraction *gś* 'half', and the *r* 'part' constituent of the general fraction numeral are masculine substantives. The feminine abstract substantive

16 Bagaria (2014); Kudryavtsev (2013).

p(s)š.t 'share', 'portion', 'part' – derived from the verb *psš* 'to distribute' – became to be used with the meaning 'half'.

A fraction numeral selects an argument encoding the number over which the fraction operates. The argument is related to the fraction by means of genitival syntax:

(19a) *gś* *n* *rmṯ.w*
half(M)[SG] GEN man(M).PL
'half of the men'

(19b) *gś* *n* *r.wj*
half(M)[SG] GEN part(M).DU
'half of two thirds'

We will focus on the syntax of the fraction and its argument in next section.

The masculine *gś*-fraction 'half' (Osing 1976: 221) is used in its singular form and it denotes a unique element (Erman & Grapow 1940). The *gś*-fraction describes a property which applies to a singleton set: Only one thing has the property of being 'half' of an element. The meaning of the *gś*-fraction could be described as:

(20) 'The *gś* fraction denotes the unique part returned from a halving operation.'

The paraphrase above characterises the meaning of *gś* 'half' as a computational property, which was derived as an extension of the natural meaning of the word *gś* used to express 'side' or 'middle':

(21a) *gś* *rmṯ*
half(M)[SG] man(M)[SG]
'half a man'

(21b) *ḥr* *gś=f*
on N(M)[SG]=3MSG
'on his side'

(21c) *gś* *ỉtrw*
N(M)[SG] N(M)[SG]
'side of a river'

(21d) *gś* *wꜣt*
N(M)[SG] N(M)[SG]
'middle of the path'

In (21a) *gś* denotes one of two parts which make the body of a person complete. In (21b), (21c) and (21d), this word is used with a locative meaning.

The computational property denoted by the natural fraction *gś* as the part that results from dimidiating generalises to include an operation of division by any number. And hence, any fraction with the form *r*-num is characterised as a numeral that denotes the *n*th part obtained by a partitioning operation:

(22) 'A "*r*-NUM" fraction denotes the unique part returned from a division (*psš*) operation.'

The computational nature of Egyptian unit fractions is manifested by the Old Egyptian denominations. The verb *ḥśb* meaning 'fractioning' and also 'reckoning' named the metrological fraction 'quarter' of a *śṯꜣt* (cf. section 2.1, table 4). The substantive *r* 'part' named the class of elements (fractional part) yielded from dimidiating the natural fraction *r.wj* '2 parts of 3'. The words *ḥśb* and *r* named thus the fractions *per definitionem* (Loprieno 1986: 1307), which referred to the operation and the part returned from applying it to a number. The old names *ḥśb*, *r* were later substituted by *r-fdw* (‘quarter’), and *r-ẖmtw* (‘third’), the general denomination of fractional numerals built on the ‘part’ concept.[17]

The “*r* NUM” fraction numeral is a uni-number nominal construction formed with the singular masculine substantive *r* and a number word which in its ideographic form occupies a post-nominal position. The word *r:wj* expressing the natural complementary fraction ‘two parts’ is inflected with dual number. The number word *r:wj* encodes a bi-number nominal with the meaning ‘2 parts of a total of 3’. The number word denoting the total number of 3 parts remained implicit.[18]

(23a) *r* *ẖmtw*
part(M)[SG] 3(UF)
3̇
‘third part’

(23b) *r:wj*
part(M).DU(CF)
3̈
‘2 parts (of a total of 3)’

The semantic contribution of the substantive *r* in both, unit and complementary fractions, was to restrict the domain of the numeral to the set of ‘fractional parts’ (Sethe 1916). The variable number component in the unit fraction “*r*-NUM” expresses a divisor argument in a partitioning operation.[19] In the *gś* fraction the divisor ‘2’ is encoded as a lexical feature of the root. The number component pointed to the last part returned from such a division. Gardiner describes that meaning as follows:

> “For the Egyptians the number following the word *r* had ordinal meaning; *r*-5 means ‘part 5’, i.e. ‘the fifth part’ which concludes a row of equal parts together constituting a single set of five. As being the part which completed the row into one series of the number indicated, the Egyptian *r*-fraction was necessarily a fraction with [...] unity as the numerator. To the Egyptian mind it would have seemed nonsense and self-contradictory to write *r*-7 4 or the like for 4⁄7; in any series of seven, only one part could

17 In pDemotic (Parker 1972), the substantive *rꜣ* became to mean not only ‘fractional part’, but also ‘fraction number’ and ‘quantity’ as did *pꜣ nkt*, which meant ‘number’ (pDemotic 8:10; 7:19):

(ia) pDemotic 3:5
pꜣ *rꜣ*
ART.M.SG N(M)[SG]
‘the fraction’

(ib) pDemotic 7:19
pꜣ *nkt*
ART.M.SG N(M)[SG]
‘the number’

18 No other complementary fraction is used in pRhind. The old ‘three quarter’ complementary fraction used in the metrological length system did not seem to be a basic one in Egyptian mathematics as was *r:wj* ‘two parts’, which generated the series 2⁄3, ⁄3, ⁄6, ⁄12, etc. Demotic papyri uses in addition of 2⁄3, the complementary fraction 5/6: pDemotic 50:16, 17 (Parker 1972) dated from Ptolemaic times 3rd century BC.

19 That is, the number over which to operate (*wꜣḥ*) in order to find (*gm.t*) the dividend (cf. section 2.3: ex. 8b).

be the seventh, namely that which occupied the seventh place in the row of seven equal parts laid out for inspection." (Gardiner 1957: §265:196)

The components of a unit fraction numeral – the singular substantive *r* and the number – form a compound. The distribution of pronominal suffixes in "*r*-NUM" favours a compound analysis. In statements in which the argument quantified by the fraction is expressed by a pronominal affix, the pronominal appears suffixed to the number component of the fraction (24a) and not to the substantive (24b):

(24a)	*r*	*fdw=f*
	part	4(UF)=3MSG
	'its fourth part'	

(24b)	#*r=f*	*fdw*
	part=3MSG	4(UF)
	'its part fourth'	

The distribution of pronominal affixes in the "*r*-NUM" construction contrasts with that of the nominal construction "*sp.w* NUM" '*n* times' expressing a multiplicative numeral (pRhind 61b:L4, in examples (50a) (50b), repeated below to ease readability):

(50a)	*sp=f*	*2*
	time=3MSG	2
	'2 times it'	

(50b)	*sp.w*	*6=f*
	time.PL	6=3MSG
	'6 times it'	

The constituents in multiplicative numerals do not seem to form a compound and a dependent pronominal may be suffixed either to the substantive (50a) or to the numeral (50b).

Another piece of evidence comes from Demotic unit fraction expressions formed with the feminine substantive *dnỉ.t* denoting 'share' 'part'. In those constructions, the compound "*r*-NUM" appears juxtaposed after *dnỉ(.t)*, a syntactic distribution also shared by the simple 'half' denoting N *gś* or *pš(.t)*:

(25a)	*tꜣ*	*dnỉ.t*	*r-dj*
	ART.F	part.F	part(M)-5(UF)
	'the share 5th-part'		

(25b)	*tꜣ*	*dnỉ.t*	*pš*
	ART.F	part.F	half(F)
	the	part	half
	'half'		

The compund nature of the noun denoting 'part' and *pš* becomes evident in the demotic compound expression *tny(.t)-pš* (Johnson 2012, 12.1:243) 'half share, division'.

In sum: Unit fractions (*gś*, "*r*-NUM") express a number by referring to the unique part returned from division – a quotient. The natural fraction *gś* 'half' encodes the divisor as a lexical feature. The fraction "*r*-NUM" used as the general notation for unit fractions is a compound. It expresses the divisor by means of the number which follows the substantive *r*. The divided argument is conveyed by a number denoting nominal category related to a fraction by means of genitival syntax.

Those lexical and selectional properties of Egyptian unit fractions could be represented in a Minimalist Syntax model by means of features.

(26) Features of unit fractions

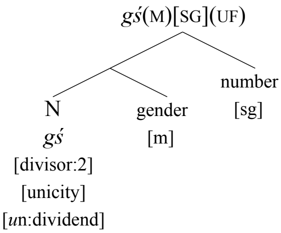
(a) Features of *gś* [num:singular], [unicity], [*u*n:dividend]

(b) Features of *r* [num:singular], [unicity], [*u*n:divisor], [*u*n:dividend]

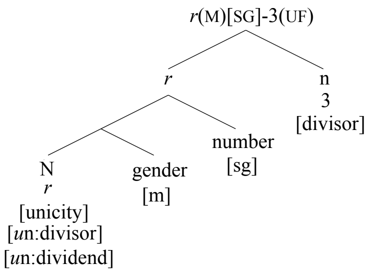
Both unit fractions *gś*, *r*-NUM start the derivation with the interpretable features singular number and unicity. The fraction *gś* comes in the lexicon with an interpretable number feature [divisor:2]; an uninterpretable [*u*n:dividend] number feature stands for the selectional properties. The substantive *r* is marked with two uninterpretable number selectional features [*u*n:divisor] [*u*n:dividend].

The syntactic structure for *gś* and *r*-3 would be the one represented in the trees below:

(27)



(28)



Middle Egyptian substantives were synthetic in form. The grammatical categories encoding gender and number – when overt – were merged with the root substantive. Unicity meaning was drawn from the interpretation of unit fractions as quotient. Unicity was derived from morphological singular number. A pronominal affix referring to the selected argument standing for a dividend was suffixed to the (compound) nominal (*gś=f*, *rꜣ-ḫmtw=f*) (cf. section 3.2). In the analytical syntactic structures of late Egyptian, the grammatical categories informing about gender and number of the substantive and unicity of unit fractions were expressed by the determiner. And the pronominal affix – which was suffixed to *gś* or to *r*-NUM in the synthetic forms – became suffixed to the determiner in the analytical forms (Depuydt 1999, Kammerzell 2000, Loprieno 1995):

(29a) pDemotic 42:6

ỉw=k	*ṯꜣ*	*tꜣy=f*	*pš*
PCL=2SG	take	POSS.ART.F=3MSG	half(F)

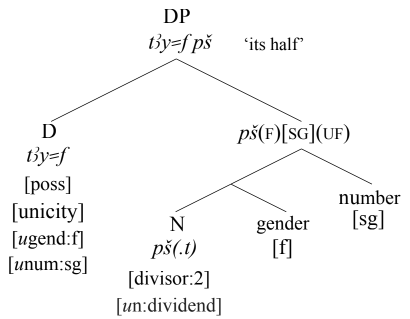
'You shall take its half'

(29b) pDemotic 46:3

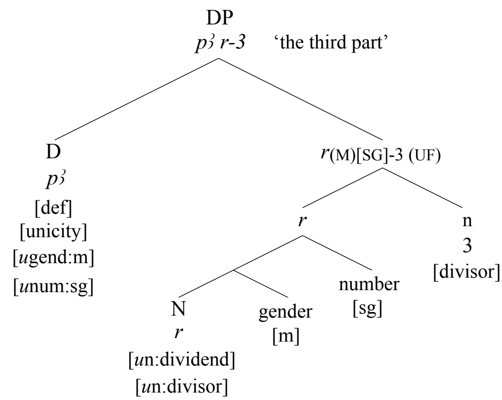
ỉw=k	*ṯꜣ*	*pꜣy=f*	*gś*
PCL=2SG	take	POSS.ART.M=3MSG	half(M)

'You shall take its half'

(30)



(31)



Complementary Fractions. The only complementary fraction documented in pRhind was *r.wj* 'two parts' '⅔'. The complementary fraction *r.wj* was linguistically encoded as a partitive construction meaning '2 parts (out of 3)'. The fraction *r.wj* did not express a unique fractional part returned from division, but a number *n* of (fractional) parts that – together with the part yielded from division – complete a (divided) whole. Complementary fractions were the precursors of Demotic bi-number fraction expressions (Sethe 1916: 61–62) such as the one illustrated below taken from Clarysse (2009: 72):

(32) A marriage contract of 187/186 BC
Ashm. dem. 7 + 8; 11 + 12+ 13: L 2

dnỉ.t	*3.t*	*ẖn*	*5.t*	*(n)*	*pꜣ*	*ꜥ.wy*
N.F	3(card).F	P	5(card).F	GEN	ART.M.SG	house

'3 parts out of 5 of the house'

The example below taken from Sethe (1916: 62) illustrates, for the fraction 'quarter', the old construction "*r-fdw*" 'the fourth part' and a new one "*1 ẖnw 4*" 'one of four' origin of 'one-fourth':

(33) Kairo 30 612b,2

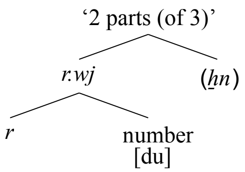
tꜣy=n	*dnỉ.t*	*r-4*	*ntj*	*ỉr*	*dnỉ.t*	*1.t*
POSS.ART.F=1PL	part.F	part-4(UF)	which	makes	part.F	one.F
ẖnw	*dnỉ.t*	*4.t*	*pꜣ*	*ꜥ.wj*		
out_of	part.F	4.F	ART.M	house		

'Our fourth part which makes 1 part out of 4 parts of the house.'

We propose for Middle Egyptian complementary fraction *r.wj*, *3̈*, 'two parts (of three)' '⅔' the syntactic structure represented in the tree below:

(34a) *r.wj*
part.DU(CF)
'two parts (of three)'
'⅔'

(34b)



The Number Phrase hosting *r.wj* would merge with a prepositional phrase containing a numeral (*ẖn* 3), which expresses the set of total fractional parts. The prepositional phrase was not overtly expressed in Middle Egyptian. We find the number phrase expressing the total number of parts in the Demotic example in (32) introduced by the preposition *ẖn*.

In the next section we study the syntax for the fraction and the argument being operated over.

3.2 The genitival constructions "*gś n r.wj*", "*gś n ỉwꜣ.w*", "*r- fdw=f*"

A fractional numeral is related to the argument affected by fractional partitioning – either a number or an entity-denoting DP – by means of genitival syntax (Sethe 1916). The fraction and its argument form a genitive construction (Schenkel 1963; Gardiner 1957: §85–86; Kammerzell 2000). The argument which the fraction applies to may be linked to the fraction by means of *n* genitival marker ("*gś n r.wj*" 'half of 2/3' pRhind 61; "*gś n prw*" 'half of the difference' pRhind 64), or it may be referred to by a pronoun suffixed to the fractional numeral ("*r-fdw=f*" 'its quarter' pRhind 61:L 10; "*r-fdw=śn*" 'their quarters' pRhind 26). We address the following questions:

- What is the syntactic distribution of those genitival constructions – the one with a genitive marker *n*, and the one with a suffixal pronoun?
- How is the operation performed by a fraction over another number syntactically encoded through those genitival structures in the language of Middle Egyptian mathematical papyri?

3.2.1 The "fraction *n*-Genitive" construction

The *n* genitival construction is used when the argument of fractional partitioning is expressed by a lexical NP or a numeral. The argument being partitioned is introduced by *n*-GEN and becomes syntactically dependent on the fraction head. The examples in table 9 illustrate the genitival construction with *n*-GEN introducing a number (*gś n mḏw* 'half of 10'), a lexical substantive (*n ỉwꜣ.w* 'cattle'), a NP formed with two substantives in apposition (*n tj.t gb.t* 'sign weak'), an interrogative pronoun (*n m* 'what'), a cardinal followed by an anaphorically used demonstrative (*n mḏw pn* 'this 10') and a fraction (*n r.wj* 'two parts').

FRAC *n*-GEN	source
gś n mḏw 'half of 10'	pRhind 52
r.wj n r-ḫmtw n ỉwꜣ.w '⅔ of ⅓ of the cattle'	pRhind 67
r.wj n tj.t gb.t '⅔ of a sign weak'	pRhind 61b
r.wj r-mḏw n m '⅔ of ⅒ of what'	pRhind 30
r-mḏw n mḏw pn '⅒ of this 10'	pRhind 28
gś n r-wj 'half of ⅔'	pRhind 61

Table 9: The "fraction *n*-genitive" construction

What is the syntactic structure encoding the mathematical operation *½ n* 'half a number' expressed by a genitival construction such as "*gś n* NUM"? We propose that the genitival construction "*gś n* NUM" expressing the operation of a fraction over a number has the syntactic structure represented by the tree (35b) below:

(35a) pRhind 52

gś	*n*	*mḏw*	*m*	*djw*
half	of	10(CARD),	P	5(CARD)

'half of 10, namely 5'

(35b) Quotient *m 5*

gś *n 10*

n *10*

The number acting as argument (10) affected by the halving operation in the example above is c-commanded by the fraction *gś* that modifies the value of the initial number by a dimidiating operation. The word *gś* denotes a function that is applied to the number 10 and returns half that number. The number 10 is a dividend. The result is introduced by *m*.

This structure is also shared by the complementary fraction *r.wj*, *3̈*, ‘two parts’ ⅔:

(36a) pRhind 61b

r.wj	*n*	*tj.t*	*gb.t*
part.DU(CF)	GEN	N.F[SG]	A.F[SG]

‘two parts of a symbol weak’
‘⅔ of a fraction with an odd denominator’
‘⅔ (1/x^{odd})’

(36b) *r.wj n tj.t gb.t*

r.wj *n tj.t gb.t*

n *tj.t gb.t*

3.2.2 The “fraction=*pro*” construction

The argument over which a fraction operates may be linked to the fraction encoded as a pronominal suffix. A lexical NP or number word acts as antecedent of the pronoun. The pronominal construction is used when the fraction and the argument – expressed by a lexical NP or number word – are not constituents of the same syntactic phrase. Between the syntactic phrase including the lexical category acting as antecedent and the phrase including the fraction and a pronominal mediate some computational steps, which are required to solve some problem. We consider three of such cases that illustrate some computational properties of pronominal fractional structures:

a) The pronominal suffix has as its antecedent a noun denoting some unknown quantity (the masculine “*pꜣ ꜥḥꜥ*” ‘the quantity’):

$\boxed{ꜥḥꜥ_i}$ ...FRAC=pro_i

b) The pronominal suffix refers to a number word antecedent:

$\boxed{r\text{-}djw_i}$, $\dot{9}{=}f_i$

c) The pronominal suffix refers to several possibly different numbers yielded from some intermediate operations.

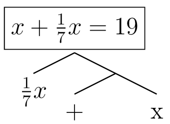
3.2.2.1 The construction "ꜥḥꜥ$_i$...FRAC=*pro*$_i$"

The argument over which a fraction operates is the masculine substantive *ꜥḥꜥ* meaning 'quantity', which acts in those contexts as a variable in the domain of numbers. The noun *ꜥḥꜥ* is not directly merged with the fraction but it occupies a left-periphery position from where it serves as the antecedent of a co-indexed pronoun suffixed to the fraction:

(37) pRhind 24:1

ꜥḥꜥ	*r-sfḫḫw=f*	*ḥr=f*	*ḫpr=f*	*m*	*psḏjw*
N(M)[SG].ANT	part-7(UF)=3MSG	P=3MSG	V=3MSG	P	19(CARD)
quantity,	7th part of it	added to it	it becomes		19

'the addition of some quantity to the seventh part of it yields 19'



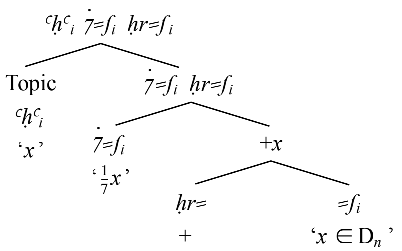
We propose that the left-periphery position occupied by *ꜥḥꜥ* – the unknown quantity – in the statement above is a topic position. The substantive *ꜥḥꜥ* referring to a variable c-commands the two instantiations of the pronominal suffix (*=f*) co-indexed with it:

(38)

ꜥḥꜥ$_i$	*7=f$_i$*	*ḥr=f$_i$*	*ḫpr=f$_i$ m*	*19*
quantity,	7th part of it,	added to it	it becomes	19

We represent the structure above in tree form:

(39)



The mathematical statement of addition operating over an unknown quantity *ꜥḥꜥ* in (37) is linguistically expressed by a predicate of addition – denoted by the preposition *ḥr* – which applies to an open variable (x). The variable position in the argument slots is occupied by a pronoun suffixed to the preposition *ḥr=f* and the fraction *7̇=f*. The substantive *ꜥḥꜥ* restricts the domain D of the variable x to numbers ($x \in D_n$). The value of the variable remains unknown.

3.2.2.2 The construction "NUM, FRAC=*f*"

The number functioning as argument affected by the fractioning operation is encoded as a pronoun suffixed to the fraction. The word denoting a number, which is the antecedent of the pronoun, is located to the left of the fraction outside its syntactic phrase:

(40) NUM$_i$,… [… FRAC=f_i …]

There is one context with a minimal pair of genitival constructions where the choice of *n* genitival construction or pronominal construction seems to be constraint by the nature of the operation being performed by the fraction. In the table of multiplication of fractions in pRhind 61, we find two different forms of stating multiplication of fractions: one using *n*-GEN (Lines 1–4) and another using a pronominal suffix (Lines 10–14):

(41a) pRhind 61:1
$\ddot{3}$ *n* $\ddot{3}$ *m* $\dot{3}$ $\dot{9}$
2(CF) GEN 2(CF) P 3(UF) 9(UF)
'⅔ of ⅔ is ⅓ + ⅑'

(41b) pRhind 61:12
$\dot{7}$ $\dot{2}$=*f* *m* $\dot{1}4$
7(UF).ANT 2(UF)=3MSG P 14(UF)
'⅐, ½ of it is 1⁄14'

Moreover, in line 9, both constructions are used:

(42a) pRhind 61:9
$\dot{9}$ *n* $\ddot{3}$ *m* $\dot{1}8$ $\dot{5}4$
9(UF) GEN 2(CF) P 18(UF) 54(UF)
'⅑ of ⅔ is 1⁄18 + 1⁄54'

(42b) $\dot{9}$ $\ddot{3}$=*f* *m* $\dot{1}8$ $\dot{5}4$
9(UF).ANT, 2(CF)=3MSG P 18(UF) 54(UF)
'⅑, ⅔ of it is 1⁄18 + [1⁄54]'

The contrast in this minimal pair of statements was noticed by Peet (1923: 103). Peet suggested that the structure (41a) with *n*-GEN seems to be used when the fraction is a proper multiplier used in Egyptian mathematics. The fractions that can act as Egyptian legitimate multipliers or divisors are ⅔ and ½ and a fraction of those obtained by halving (⅔, ⅓, ⅙, 1⁄12...; ½, ¼, ⅛...). The structure (41b) with the pronoun is used when the fraction is not a proper multiplier. Peet (1923) makes the following considerations concerning the duplicity of statements in L9:

> "An Egyptian cannot take one-ninth of ⅔: he can take one-third of any quantity by simply taking two-thirds and halving it, but he cannot obtain one-ninth direct from one-third, for he cannot divide by 3, only by 2. The consequence is that to speak of taking one-ninth is technically incorrect, and the Egyptian should avoid the use of the phrase even in a table of results. Thus in line 9 we find in the column the correct

form of statement: “One-ninth, ⅔ of it is 1⁄18 + 1⁄54.” The less correct “One-ninth of ⅔ is 1⁄18 + 1⁄54” being added in the margin owing to an error explained below. It would seem that the succeeding lines of the table were written in the correct form, but without the marginal addition.” Peet (1923: 103–104)

The construction with pronominal genitive seems, thus, to be used when the operation is not an immediate one, but it requires some intermediate steps to be performed.

3.2.2.3 The pronominal suffix refers to some number obtained from reckoning

In pRhind 67 there is another example where both the structures *n*-GEN and pronominal genitive are used:

(43a) pRhind 67:3–4

r.wj	*n*	*r-ẖmtw*	*n*	*ỉwꜣ.w*
2(CF)	of	3(UF)	of	cattle

‘⅔ of ⅓ of the cattle’

(43b) pRhind 67:8

r.wj	*n*	*ẖmtw=f*
2(CF)	of	3(UF)=3MSG

‘⅔ of ⅓ of it’

The antecedent of the masculine singular 3rd person pronominal suffix *f* is a number.[20] More precisely the pronoun refers to two numbers of different cardinality value: 1 in L 5 and 315 in L 18:

(44)

ỉwꜣ.w$_i$ ...	*1*$_i$	*...315*$_i$	*..f*$_i$
cattle.ANT	1(CARD).ANT	315(CARD).ANT	3MSG

‘3 parts (of 4)’

Both those numbers are obtained by reckoning. The computational steps are:

Operation	Argument
1	$\boxed{1}$
$\ddot{3}$	$\ddot{3}$
$\dot{3}$	$\dot{3}$
$\ddot{3}$ *n* $\dot{3}$= $\boxed{f_i}$	*m* $\dot{6}$ $\dot{1}8$

Figure 4: Number 1 as antecedent of the pronoun *f*

20 If it were the N *ỉwꜣ.w*, the pronoun would have been 3PL *sn* and not 3M.SG *f*.

a) ⅔ of ⅓ of 1 (*ỉwꜣ.w*) = ⅙ + ⅟18:[21] The substantive *ỉwꜣ.w* ‘cattle’ is been referred as the number 1 in the operations:

(45) pRhind 67:8
3̈ *n* *3̇=f* *m* *6̇* *1̇8*
2(CF) GEN 3(UF)=3MSG P 6(UF) 18(UF)
‘⅔ of ⅓ of it is ⅙ + ⅟18’

b) 1 divided by ⅙ + ⅟18 = 4 ½. The division operation “1 divided by ⅙ ⅟18” is expressed by the statement below:

(46) *nỉś.ẖr=k* *1* *m* *6̇* *1̇8*
V.DIR=2SG 1(CARD) P 6(UF) 18(UF)
‘call 1 out of ⅙ + ⅟18. [Result 4 + ½]’

c) Multiplication of 70 by 4 + 1/2 = 315, which gives the number of cattle committed to the herdsman, and which will act as antecedent referred by the pronoun *f* suffixed to 3̇ ‘third’ in the number ‘3̈ of 3̇’:

(47) *r.wj* *n* *r-ẖmt=f*
3̈ *n* *3̇=f*
2(CF) GEN 3(UF)=3MSG
‘⅔ of ⅓ of it’

Operation	Argument
1	$\boxed{315_i}$
3̈	210
3̇	105
3̈ *n* 3̇ = $\boxed{f_i}$	70

Figure 5: Number 315 as antecedent of the pronoun *f*

(48) pRhind 67:15–16
ỉr *70* *r* *sp.w* *4* *2̇* *ẖpr* *(r)* *315*
operate_on 70 up_to times 4 1/2; becomes 315
‘Multiply 70 by 4 ½; it becomes 315’

Another example in which the antecedent of the suffixal pronoun is provided by reckoning is in pRhind 61b: In this problem the pronominal suffix *f* refers to a number 5̇ (the fraction ⅕) and to an NP denoting a number concept “*tj.t gb.t*” (‘sign uneven’, ‘weak symbol’: the reciprocal of an odd number):

21 Accordingly to the common practice in Egyptian Mathematics, the result is expressed as a sum of unit fractions and not by a common fraction.

(49a) pRhind 61b:3
pty *3̈* *n* *5̇*
PROINT 2(CF) of 5(UF)
'What is ⅔ of ⅕?'

(49b) pRhind 61b:1
3̈ *n* *tj.t* *gb.t*
2(CF) of N.F A.F
'2 parts of sign uneven'
'⅔ of the reciprocal of an odd number'

The masculine 3rd person pronoun *f* is suffixed to (a) a multiplier numeral; (b) the fraction *r:wj* ⅔:

a) The pronoun is suffixed to a multiplier numeral, either to the noun *sp*: "*sp=f 2*" (2 times it:5), or to the number component in the multiplier numeral: "*sp.w 6=f*" (6 times it:5). In this case, the pronoun *f* refers to the reciprocal of the fraction ⅕ (pRhind 61b. L 4):

(50a) *sp=f* *2*
time=3MSG 2
'2 times it (it=5)'

(50b) *sp.w* *6=f*
times 6=3MSG
'6 times it (it=5)'

b) The pronoun is suffixed to the fraction 3̈=*f*, when the result is stated. Here the pronoun *f* refers to the fraction ⅕:

(51) pRhind 51b:5
3̈=f *pw*
⅔=3MSG (5̇), this is
'this is: ⅔ of it (it=⅕)
is ½x5 + ⅙x5 = ⅒ + 1⁄30; i.e.: the reciprocals of (2x5) + (6x5)'

The use of the 3rd person masculine pronoun *f* to refer to an abstract variable number rather that to an entity of the World described by a lexical NP or to a constant referring to a metrological unit evidences that fractions were treated as a class of numbers referentially independent. In the example below from pDemotic, the pronominal suffix in the possessive article acting as determiner of a fraction refers to a DP with a substantive denoting a 'number' concept and not to a world dependent property:

(52) pDemotic 42:10, 16
pꜣ *r* *...pyꜣ=f* *2̇* *m* *x*
the number$_i$ …its$_i$ half is *x*

4 Conclusions

The aim of this article has been to study the lexical and grammatical categories encoding Egyptian fractional numerals, and the syntactic operations used for relating a fraction with its argument. We have focused on data from Middle Egyptian mathematical papyri (pRhind).

Egyptian fractional numerals are partitive nominal expressions of two types: a simple substantive *gś* 'half', *r.wj* 'two parts (of three)', and a complex nominal *r*-NUM 'the *n*th part', which is the general notation for fractions. Partitivity is encoded through the lexical and selectional features of *gś* and *r*. Those substantives select two number arguments: a divisor and a dividend. The divisor argument of *r* is expressed by a variable number component: the two constituents form a uni-number numeral *r*-NUM. The divisor in *gś* is a constant number '2' encoded as a lexical feature. The dividend argument is expressed by a variable numeral or entity denoting DP related to the fraction by means of genitival syntax: It may be introduced by genitival *n* when lexical (*gś n mḏw* 'half of 10') or referred to by a pronominal suffixed to the fraction (*ꜥḥꜥ, r-sfḫw=f* 'a number, the seventh part of it'). The expressions *gś*, *r*-NUM, with the substantive in singular number, are unit fractions. They denote a unique element which is the part returned from division (*ḥśb*, *psš*). The numeral *r.wj*, with the substantive in dual number, is a complementary fraction. It refers to the two parts that together with the third part make a whole. The complementary fraction is a bi-number construction that expresses a ratio between the complementary parts and the total number of parts, which remain implicit.

Middle Egyptian fractional numerals provide us with pieces of evidence of the chain of linguistic and social factors contributing to the origin and development of the concept of fractional number. Due to the extended period of time of the use of the Egyptian language, and the influence that Egyptian fractions had on the linguistic numeral systems developed by other cultures and languages, the study of Egyptian fractions is an important domain of inquiry for linguistic research.

Abbreviations

1 = first person
2 = second person
3 = third person
ANT = antecedent
ART = definite article
CARD = cardinal
CF = complementary fraction
DEM = demonstrative
DIR = directive
DU = dual number feature
F = feminine gender suffix
f = feminine gender feature
FRAC = fraction
GEN = genitive
m = masculine feature
M = masculine gender suffix
NUM = numeral
PCL = particle
PL = plural
POSS = possessive
PROINT = interrogative pronoun
RES = resultative
sg= singular number feature
SG = singular
TOP = topic
UF = unit fraction

Glosses follow the Leipzig Glossing Rules:
http://www.eva.mpg.de/lingua/resources/glossing-rules

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